Short Note

A co-located incompressible Navier–Stokes solver with exact mass, momentum and kinetic energy conservation in the inviscid limit

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1. Introduction

The Navier–Stokes equations conserve mass, momentum and kinetic energy in the limits of inviscid and incompressible flow. The equation for kinetic energy is derived from the momentum equation; it is therefore a consequence of the discretized momentum balance rather than a separate equation. For this reason, the conservation of kinetic energy is commonly referred to as secondary conservation, in contrast to the primary conservation of mass and momentum. When solving the Navier–Stokes equations numerically, there are several reasons to discretely mimic the exact conservation properties of these three quantities. First, since the kinetic energy is a norm of the solution, a method that conserves this property is guaranteed to be stable. Secondly, it is well known that absence of artificial dissipation leads to vastly improved accuracy in large eddy simulations (LES) [1].

The discrete conservation properties of numerical methods for the Navier–Stokes equations are strongly dependent on the way variables are arranged on the grid. On staggered grids, discrete operators based on symmetric central difference stencils with primary and secondary conservation have been constructed in several studies [2–5]. For complex solution domains, a co-located variable arrangement holds significant advantages over the staggered one. On co-located grids, however, the use of symmetric central difference operators gives rise to the problem commonly known as pressure checker-boarding [6]. The pressure checker-boarding arises due to the resulting wider stencil of the Laplacian in the pressure equation, which decouples nearby grid points. This problem of pressure checker-boarding is widely known, and various workarounds have been proposed in the literature [7–10]. These cures are similar in the sense that they all introduce a face velocity that (explicitly or implicitly) depends on an interpolated pressure gradient. Although this eliminates the problem of pressure checker-boarding, it introduces a spurious term in the pressure equation that causes dissipation of kinetic energy [6,11,12]. As a

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result, a solver with co-located variable arrangement that exactly conserves mass, momentum and kinetic energy has not yet been formulated in the literature.

The objective of the present paper is to develop a solution procedure for the incompressible Navier–Stokes equations with co-located variable arrangement that exactly conserves mass, momentum and kinetic energy in the inviscid limit. This is accomplished by utilizing vectors that span the null space of the discrete pressure Laplacian to obtain a smooth pressure field. The method is developed here in the context of Cartesian grids to illustrate the concept, but it should generalize to more complex mesh arrangements.

2. Governing equations

The governing equations for incompressible flow in Cartesian coordinates are

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{1a}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j}, \tag{1b}$$

where P is the pressure divided by the constant density, u_i is the velocity vector, and the viscous stress tensor is

$$\tau_{ji} = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

where ν is the kinematic viscosity. The Poisson equation for pressure is derived by taking the divergence of (1b) and using (1a); this yields

$$\frac{\partial^2 P}{\partial x_i^2} = -\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j}. \tag{2}$$

The incompressible Navier–Stokes equations are insensitive to the mean level of pressure; this is reflected mathematically by the Laplacian $\partial^2/\partial x_i^2$ in (2) being singular with a null space of rank 1 corresponding to constant functions.

3. Semi-discrete equations on a Cartesian grid

Denote a second-order accurate central difference approximation of the first derivative in the 1-direction by

$$\left.\frac{\delta f}{\delta x_1}\right|_{(ijk)} = \frac{f_{(i+1,j,k)} - f_{(i-1,j,k)}}{2\Delta x_{1,(i)}},$$

where subscripts in parentheses indicate the index of the grid point and x_1, x_2, x_3 are the coordinate directions. Note that Δx_1 only depends on the (i) index for an orthogonal structured (but possibly stretched) Cartesian grid. To discretely conserve energy the convective term is discretized in its skew-symmetric form; this yields the semi-discrete equations for conservation of mass and momentum as

$$\frac{\delta u_i}{\delta x_i} = 0, \tag{3a}$$

$$\frac{du_i}{dt} + \frac{1}{2}\frac{\delta u_i u_j}{\delta x_j} + \frac{1}{2}u_j\frac{\delta u_i}{\delta x_j} = -\frac{\delta P}{\delta x_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2}, \tag{3b}$$

where the viscous term is left in continuous form since this work only considers the limit of inviscid flow. It is straight-forward to show that Eqs. (3a) and (3b) discretely conserves mass, momentum and kinetic energy on a periodic domain. The discrete Poisson equation for pressure is then derived by applying the discrete divergence operator $\delta/\delta x_i$ to (3b) and making use of (3a); this yields

$$\frac{\delta}{\delta x_i}\frac{\delta P}{\delta x_i} = -\frac{1}{2}\frac{\delta}{\delta x_i}\left(\frac{\delta u_i u_j}{\delta x_j} + u_j\frac{\delta u_i}{\delta x_j}\right). \tag{4}$$

The discrete Laplacian is the product of two discrete first derivatives, which implies that its null space includes odd–even oscillations (the π-mode) in addition to constant functions. For example, in 1D the discrete Laplacian with a second-order accurate scheme is

$$\frac{1}{2\Delta x_{1,(i)}}\left(\frac{P_{(i+2,jk)} - P_{(ijk)}}{2\Delta x_{1,(i+1)}} - \frac{P_{(ijk)} - P_{(i-2,jk)}}{2\Delta x_{1,(i-1)}}\right), \tag{5}$$

which is simply an approximation to the second derivative on a twice coarser grid. Higher-order schemes yield a similar result. In multiple dimensions, the null space trivially includes all modes that are either constant or odd-even oscillations in every direction. To formalize this, the null space of the discrete 3D Laplacian in (4) is spanned by the modes

$$\widehat{P}_{0,(ijk)} = 1, \quad \widehat{P}_{1,(ijk)} = (-1)^i, \quad \widehat{P}_{2,(ijk)} = (-1)^j, \quad \widehat{P}_{3,(ijk)} = (-1)^{i+j}, \quad \widehat{P}_{4,(ijk)} = (-1)^k, \quad \widehat{P}_{5,(ijk)} = (-1)^{i+k},$$
$$\widehat{P}_{6,(ijk)} = (-1)^{j+k}, \quad \widehat{P}_{7,(ijk)} = (-1)^{i+j+k}$$

In other words, if $\tilde{p}$ is a solution of (4), then

$$P_{(ijk)} = \tilde{p}_{(ijk)} + \sum_{l=0}^{7} a_l \widehat{P}_{l,(ijk)} \tag{6}$$

for any vector $\mathbf{a} = (a_0, a_1, \ldots, a_7)^T$, is also a solution of (4). Note that these specific vectors span the null space only for a Laplacian operator on a Cartesian grid and this is the only step in the algorithm that depends on the grid connectivity and the discretization method. In the general case it is always possible to construct null space vectors using an algorithm like singular value decomposition (SVD) [13].

The approach taken in this work is to first solve the discrete Poisson equation, and then to modify the obtained solution $\tilde{p}$ by adding some combination of null space modes to create a smooth final pressure field. In principle, this is no different from the standard practice in incompressible flow solvers to add an arbitrary mean pressure field afterwards; it is simply a reflection of the fact that the Laplacian is singular. This can also be viewed as a special kind of filtering of the pressure field, where the filtering is restricted to the null space of the Laplacian to preserve kinetic energy conservation. Finally, the common practice of adding Rhie–Chow [7] dissipation to the system can be seen as a way to modify (4); since the Rhie–Chow method operates outside of the null space, it destroys the kinetic energy conservation property.

It is clear that the final pressure field P will solve the discrete Poisson equation provided that (6) is used to ensure smoothness; the question is how should the coefficients $\mathbf{a}$ be chosen? One natural approach is to measure the (lack of) smoothness of P in the L_2 norm, and require this to be minimum. This is appealing from a physical point-of-view, and is what is implicitly done in Rhie–Chow regularization. In vector notation the weighted L_2 norm is given by

$$\|P\|^2 = P^T V P,$$

where P is a vector with elements $P_{(ijk)}$ in some ordering and V is a symmetric positive definite matrix. In the present work we simply take V to be diagonal with elements $\Delta x_{1,(i)} \Delta x_{2,(j)} \Delta x_{3,(k)}$, i.e., the volume of each grid cell. Inserting expression (6) yields a convex quadratic

$$\|P\|^2 = \left(\tilde{p} + \sum_l a_l \widehat{P}_l \right)^T V \left(\tilde{p} + \sum_l a_l \widehat{P}_l \right),$$

with the minimizer given by

$$\mathbf{a} = - \begin{pmatrix} \widehat{P}_0^T V \widehat{P}_0 & \widehat{P}_0^T V \widehat{P}_1 & \cdots & \widehat{P}_0^T V \widehat{P}_7 \\ \widehat{P}_1^T V \widehat{P}_0 & \widehat{P}_1^T V \widehat{P}_1 & \cdots & \widehat{P}_1^T V \widehat{P}_7 \\ \vdots & & \ddots & \vdots \\ \widehat{P}_7^T V \widehat{P}_0 & \widehat{P}_7^T V \widehat{P}_1 & \cdots & \widehat{P}_7^T V \widehat{P}_7 \end{pmatrix}^{-1} \begin{pmatrix} \widehat{P}_0^T V \tilde{p} \\ \widehat{P}_1^T V \tilde{p} \\ \vdots \\ \widehat{P}_7^T V \tilde{p} \end{pmatrix}. \tag{7}$$

This is simply an 8×8 system that can be easily solved for the coefficients $\mathbf{a}$. In fact, the matrix does not depend on the solution, and can therefore be inverted in a pre-processing step.

To summarize, the proposed algorithm is to solve the Poisson equation for $\tilde{p}$, then to find $\mathbf{a}$ from (7), and then to get the final pressure field P from (6). The computational cost of the two additional steps is insignificant compared to the Poisson solve.

4. Numerical experiments

The equations are integrated in time using the Crank–Nicolson scheme. The implicit coupled non-linear system of equations are solved using the fractional step algorithm [14], with the approximate factorization technique as outlined in [15] used for implicit solution of momentum equations. In the following three inviscid test problems in periodic domains are presented, with the goal of confirming the exact conservation properties and that the order of accuracy is maintained.

4.1. Taylor vortex

This two-dimensional flow is an analytical solution of the Navier–Stokes equations [14]. In the inviscid limit, the analytical solution on a periodic domain $x, y \in [-1, 1)$ is
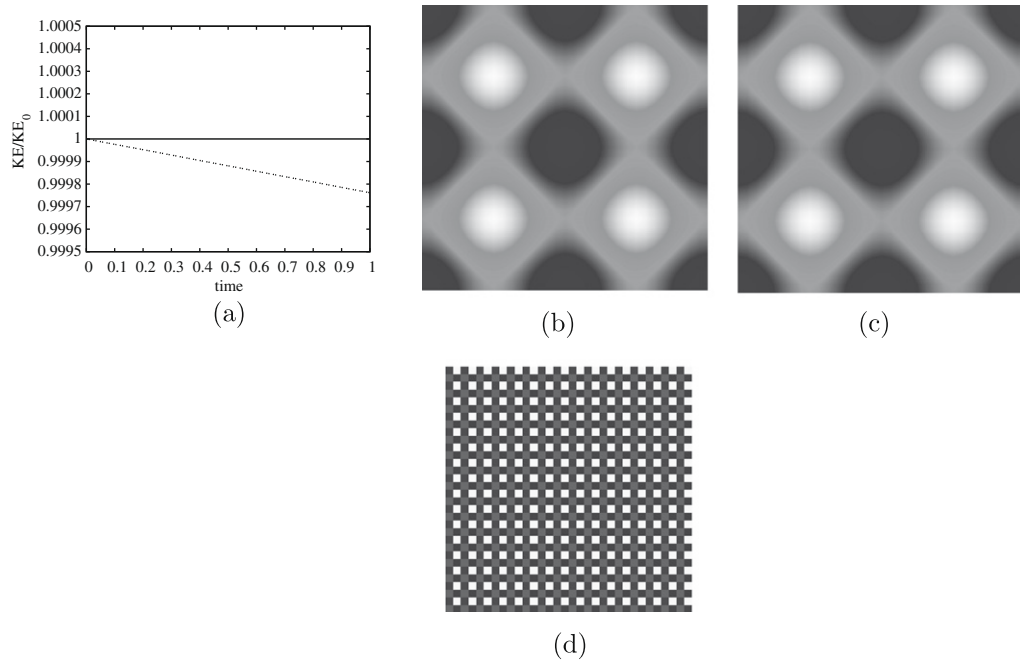
Fig. 1. Inviscid Taylor vortex. (a) Temporal evolution of kinetic energy using the present (solid) and Rhie–Chow (dotted) methods. (b–d) Pressure contours for the present (b), Rhie–Chow method (c) and without any correction (d).

$$u = -\cos(\pi x)\sin(\pi y),$$
$$v = \sin(\pi x)\cos(\pi y),$$
$$p = -\frac{\cos(2\pi x) + \cos(2\pi y)}{4}.$$

Fig. 1(a) shows the time evolution of kinetic energy on a 32^2 mesh with both the present method and when using the Rhie–Chow [7] momentum interpolation. The conservation of kinetic energy with the present method, and the lack thereof with Rhie–Chow interpolation, are clearly seen. To demonstrate the smoothness of the pressure field, Fig. 1(b) and (c) compares pressure contours from the two methods; the present method generates an equally smooth pressure field as the Rhie–Chow method. In contrast, Fig. 1(d) shows the pressure field without the null-space correction (or Rhie–Chow interpolation); the pressure checker-boarding is clearly evident.

4.2. Convecting vortex

To evaluate the order of accuracy we consider a vortex convected by a uniform flow on a domain $x, y \in [-0.5, 0.5)$. The initial velocity field is taken as [16]

$$u = U_\infty - C\frac{(y - y_c)}{R^2}e^{-r^2/2}, \quad v = C\frac{(x - x_c)}{R^2}e^{-r^2/2},$$

where $r = \sqrt{(x - x_c)^2 + (y - y_c)^2}/R$. The parameters are taken as $C = 0.00625$, $R = 0.125$, and $U_\infty = 1$. The L_∞ norm of the error after one period is shown in Fig. 2(a) for both the present and Rhie–Chow methods. The errors from the two methods are indistinguishable, and both decrease as Δx^2 as expected for a second-order method. Therefore, the null space strategy does not introduce additional error. Fig. 2(b) shows the temporal evolution of the total momentum and kinetic energy for 10 periods on a 64^2 mesh; the present method clearly conserves these properties to machine precision.

4.3. Inviscid turbulence in a periodic box

The final test of the proposed method is the computation of the evolution of inviscid turbulence on a domain $x, y, z \in [0, 2\pi)$ with 32^3 grid points. The initial condition consists of a three-dimensional solenoidal random field with energy spectrum

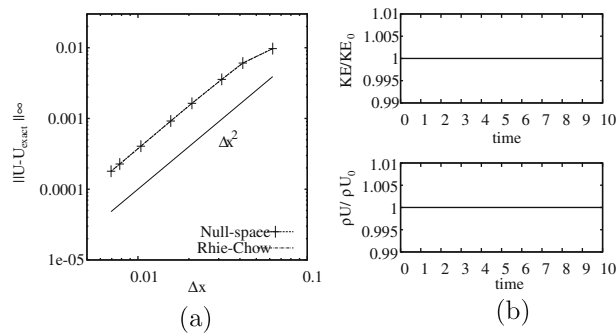$$E(\kappa) \propto \kappa^4 e^{-2\kappa^2}.$$

Fig. 2. Inviscid convecting vortex. (a) L_∞ norm of the error in u after one period. (b) Temporal evolution of kinetic energy and momentum.
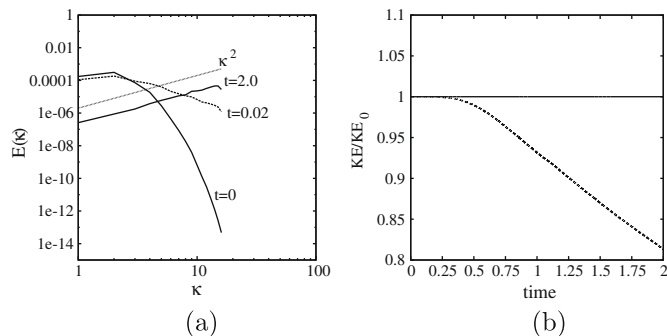

Fig. 3. Inviscid turbulence in a periodic box. (a) Temporal evolution of the energy spectrum. (b) Temporal evolution of the total kinetic energy with the present (solid) and Rhie–Chow (dotted) methods.

Fig. 3(a) shows the evolution of the energy spectrum with time. The spectrum approaches a κ^2 distribution after long time, which implies an equi-partition of energy among all wavenumbers; this is the correct behavior for this inviscid system. Fig. 3(b) shows the total kinetic energy with both the present and Rhie–Chow methods. While the present method exactly conserves the kinetic energy, the unphysical dissipation in the Rhie–Chow method decreases the energy by more than 20% over the course of the simulation.

5. Summary

A numerical method for the incompressible Navier–Stokes equations that discretely conserves mass, momentum and kinetic energy in the inviscid limit on co-located grids is presented. The method relies on symmetric central difference operators, which together with a skew-symmetric form of the convective term yields the conservation properties. The discrete Laplacian operator in the pressure equation has a null space of rank 8 in three dimensions, compared to rank 1 for the continuous equations. While existing co-located methods reduce the rank of the null space by introducing numerical dissipation (most commonly through the Rhie–Chow momentum interpolation method [7]), in the present method we retain the discrete Laplacian operator intact and utilize vectors that span the null space to achieve a smooth pressure field.

In the present note only Cartesian grids are considered, but the method should generalize to more complex geometries. Nothing in the approach inherently requires the use of structured Cartesian grids; the only challenge on more complex grids (curvilinear or unstructured) would be to find the vectors that span the null space of the discrete Laplacian.

Acknowledgements

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